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On the motion of ν-fluids

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The paper is concerned with a class of non-Newtonian fluids, ν-fluids, all of whose properties are determined by a single dimensional constant of the same dimensions as a viscosity. A regular nth-order ν-fluid is then defined to be one whose nth order time derivative of stress is a regular function of the local stress and flow fields and any of their space and time derivatives. The regularity condition determines the constitutive relation of such a fluid completely in terms of a finite set of non-dimensional constants which define the fluid.

An obvious property of these fluids is that their motions obey the same principles of Reynolds number similarity as those of a Newtonian fluid, and the primary aim of the paper is to examine the extent to which their flow properties are the same as those of the mean turbulent flow of a Newtonian fluid.

It is shown that a third-order fluid is the simplest ν-fluid which shares enough properties with turbulent motion to be worth further consideration in this context. At infinite Reynolds number, the constitutive relation for such a fluid reduces to the form

$$AS\ddot{S}+B\dot{S}^2+CS^2S''+DSS'^2+Eu'^2S^2 = 0,$$

where $A, B, \ldots, E$, are isotropic tensor constants of the fluid, S is the stress tensor, u' is the total rate of strain tensor, dots denote total time derivatives, and primes denote space derivatives. A number of illustrative examples of the properties of such a constitutive relation are then considered, representing the decay of a homogenous stress field, the effect of rigid-body rotation on such a decay, the structure of the equilibrium stress field in the presence of homogeneous rate of strain, both with and without vorticity, and the nature of flow near a plane boundary. In all cases, the results appear to be consistent with known properties of turbulent motion, to the extent that the analysis is taken.

Finally, the effect of finite Reynolds number on the decay of an isotropic and homogeneous stress field is shown to be consistent with observations on the decay of isotropic turbulence.

1. Foundations of the approach

This paper is concerned with a certain class of incompressible non-Newtonian fluids, in which the local stress system is in general not in equilibrium with the local flow. The work has its origin in a study of the turbulent motion of a Newtonian fluid, not as intended contribution to the fundamental statistical mechanics of that phenomenon (which, by its nature, the present approach cannot achieve), but as an attempt to study at a more phenomenological level the macro-properties of turbulent flow systems as a whole, and the relationships between them. By studying a class of relatively simple dynamical systems which appear to share important macro-properties with turbulent motions of a Newtonian fluid, the paper thus follows the growing trend of attempts to construct constitutive relations for turbulence, of which a recent example is the work of Lumley (1970).

A central idea in all such work is that the statistical properties of turbulent motion depend to only a limited extent on the initial and boundary conditions for the flow. The element of universality thus introduced suggests the replacement of a complicated motion of a simple (Newtonian) fluid by a simple motion of a more complicated fluid, with an attendant reduction in the technical difficulties of obtaining a solution. It is generally accepted that such an approximation, while still not necessarily good, is likely to be at its best when the Reynolds number of the turbulent flow is very large. It is therefore tempting to construct a constitutive relation only, in the first instance, for the limiting case of infinite Reynolds number. However, the dangers of discussing inviscid limits without reference to the underlying diffusive and dissipative systems are well known throughout fluid mechanics, particularly statistical mechanics, and the approach of the present paper is to start with a class of constitutive relations whose solutions for all flow configurations are well behaved provided a viscosity parameter is not zero. There is no suggestion here that any fluid of the class can represent quantitatively the turbulent motion of a Newtonian fluid at low Reynolds numbers. The approach is simply an insurance policy which restricts attention, at large Reynolds numbers, to those asymptotic constitutive relations appropriate to a class of well-behaved dynamical systems.

Thus, a characteristic of all the fluids to be considered is that their material properties involve a parameter ν, whose dimensions† are those of the viscosity of a Newtonian fluid, and which we may continue to call the viscosity. We now restrict our attention to fluids whose only dimensional physical constant is ν, and attach the name 'ν-fluids' to such fluids. The reason for this restriction is fairly clear. A Newtonian fluid is a ν-fluid, so that, if we are to represent the statistical properties of its turbulent motion by a *universal* law of stress, that law must also contain no other dimensional constant than ν. Other dimensional constants which determine the stress distribution in a particular flow must enter through the initial and boundary conditions, not through the field equations. In no other way is it possible to satisfy the well-confirmed principle of Reynolds number similarity for turbulent motion.

This emphasis on the parameter ν makes the approach of the present paper primarily suited to phenomena in which the molecular stresses of a Newtonian fluid are modified by processes which have length and time scales orders of magnitude greater than the molecular ones. Indeed, it seems likely that the approach is relevant only in such cases, which, apart from turbulence, include the

† The density of the incompressible fluid is taken to be unity, so that kinematic definitions are adopted throughout.

behaviour of homogeneous suspensions of small particles, and possibly other cases. With the usual notation, we thus represent the total stress tensor σ_{ij} by

$$\sigma_{ij} = -p\delta_{ij} + \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - S_{ij}, \tag{1.1}$$

where $-S_{ij}$ is the added stress tensor, and discuss the stress dynamics or constitutive relations entirely in terms of S_{ij}. (In the turbulence problem, $-S_{ij}$ is, of course, the Reynolds stress tensor.) An immediate and important consequence is that the absolute level of S_{ij} can be dynamically significant, while the absolute level of pressure remains arbitrary, corresponding to two entirely distinct forms of internal energy. Such a distinction is clearly important in the case of turbulence, where the Reynolds stresses arise from momentum transport by a real velocity field, so that

$$\lambda_i \lambda_j S_{ij} \geqslant 0, \tag{1.2}$$

for all vectors λ_i.

The second characteristic property of a ν-fluid is that its constitutive relation takes the form of a differential field equation of finite order in time. More precisely, a ν-fluid of order n is one in which the added stress S at a point where the velocity is u is governed, in an inertial frame of reference, by an equation of the form

$$\frac{D^n S}{Dt^n} = f\left(u, \frac{Du}{Dt}, \ldots, \frac{D^n u}{Dt^n}, S, \frac{DS}{Dt}, \ldots, \frac{D^{n-1}S}{Dt^{n-1}}, \nu\right), \tag{1.3}$$

where f denotes a function of, at most, the fields listed and any of their space derivatives of any order, and is regular at the origin of the multi-dimensional space of all these arguments. This fundamental equation (1.3) is not in any sense deductive. The appropriate viewpoint is to regard it as the definition of a restricted class of fluids with which the paper is concerned, the task then being to examine the extent to which these fluids, particularly those of low order, share macro-properties with turbulence.

The relationship between (1.3) and the classical rheological principles of determinism and material indifference perhaps deserves comment. Lumley (1970) has recently emphasized that neither of these principles can be wholly accepted in constitutive relations for turbulence, and, at this point, it may be as well to set out precisely the nature of the problem. For simplicity, we confine ourselves to cases of turbulent motion of a Newtonian fluid in which there is a unique and well-defined mean velocity u_i at every point of space-time, arising out of statistical stationarity of the total velocity distribution $u_i + u_i'$ in either time or one of the spatial co-ordinates. A similar notation for the mean (p) and fluctuating (p') pressure distribution may be introduced. Thus, the dynamical equations for this turbulent motion are

$$\frac{\partial}{\partial t}(u_i + u_i') + (u_j + u_j')\frac{\partial}{\partial x_j}(u_i + u_i') = -\frac{\partial}{\partial x_i}(p + p') + \gamma\nabla^2(u_i + u_i'), \tag{1.4}$$

and

$$\frac{\partial}{\partial x_i}(u_i + u_i') = 0. \tag{1.5}$$

In the equivalent ν-fluid problem we set

$$S_{ij} = \overline{u_i'u_j'}, \tag{1.6}$$

and thus embed the entire Reynolds-stress tensor into the material properties of the ν-fluid. In so doing, we note that the only motion in the ν-fluid is that represented by u_i, and that we shall be concerned only with laminar motion of the ν-fluid in the sense that u_i will be independent of that co-ordinate in which there is statistical stationarity. The important point now is that every subsequent statement (exact or approximate) about S_{ij}, whatever its nature, must be justified as a consequence of the dynamical system (1.4) and (1.5).

Two exact consequences of (1.4) and (1.5) are the equation of motion,

$$\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i - \frac{\partial S_{ij}}{\partial x_j}, \tag{1.7}$$

and the incompressibility condition,

$$\partial u_i/\partial x_i = 0, \tag{1.8}$$

for the ν-fluid. Another statement about S_{ij} is (1.3) itself, and, as has already been noted, the *a priori* justification in terms of (1.4) and (1.5) for such a special form of constitutive relation is extremely weak, the object of the investigation being to ascertain the extent to which an indirect *a postiori* justification is possible. The principles of determinism and material indifference, however, are not so much concerned with the restricted form of (1.3) as with further constraints which should be imposed upon it as a consequence of more general considerations, and this greater generality calls for a correspondingly greater justification in terms of (1.4) and (1.5).

Consider, for instance, that part of the principle of determinism which requires the current stress distribution to be determined by the past history of the flow. Since u_i', as determined by (1.4) and (1.5), satisfies this principle, so does S_{ij}, and this part of the principle of determinism is acceptable for a ν-fluid. Moreover, the constitutive relation (1.3), coupled with (1.7) and (1.8), already satisfies the principle. Indeed, the standard initial-value problem for an nth-order ν-fluid provides an indication of the connexion between the order n and the degree of refinement which is required or expected in the effect of externally imposed information. The situation is most easily illustrated in the case of a decaying system of homogeneous isotropic stress in a stationary fluid. In this case, (1.7) and (1.8) are satisfied identically, and (1.3) reduces to an nth-order ordinary differential equation for S as a function of time. Thus, n initial conditions are required, and the totality of solutions belong to a universal n-parameter family. In the application to turbulence, this is equivalent to requiring all energy spectra in isotropic turbulence to belong to a universal n-parameter family of functions.

The idea, also included in the principle of determinism, that the origins of stress must lie in the history of the immediate spatial neighbourhood of the fluid particle in question is, on the other hand, clearly not acceptable. Such an idea would be totally at variance with almost every aspect of fully developed turbulence. Formally, the position is that, in the system (1.4) and (1.5), the fluctuation velocity u_i' at the point x_i at time t is not determined solely by the history of the conditions in the immediate neighbourhood of that geometrical point which, always moving with the mean velocity u_i, arrives at the point x_i at time t. Hence, S_{ij} is not solely determined by these conditions. Moreover, in the equivalent ν-fluid, the geometrical point referred to becomes a material particle, so that this part of the principle of determinism is not acceptable. Nor does the constitutive relation (1.3) imply such a local determination. Nevertheless, there is *some* degree of localness in (1.3), namely, the existence of a purely local relation (the equation itself) between finite-order derivatives of the flow and stress fields. In the application to turbulence, even such a degree of localness as this must be an approximation, and the approximation represents a suggested solution to the 'closure' problem of turbulence.

The principle of material indifference asserts that the constitutive relation for any fluid is invariant with respect to arbitrary time-dependent rigid-body motions superimposed upon the flow. There is no suggestion here that the resultant motion need be dynamically possible without the simultaneous superimposition of an external force field (not necessarily conservative) to maintain it. Thus, the principle is apparently of a purely kinematical nature. The total absence of dynamics, however, is apparent rather than real, since there are implied restrictions on the nature of the superimposed external forces, these restrictions being intimately related to the dynamical substructure (molecular motion, turbulence, etc.) giving rise to the stresses.

In the case of turbulence, for instance, it is certainly possible to produce, by a suitable distribution of external forces, a motion relative to an arbitrary moving rigid frame which is identical in all its detail with a known motion relative to an inertial frame. The two motions would thus possess identical Reynolds stress distributions (taking appropriate account of direction and co-ordinate transformation between the two frames). But this trivial remark has little to do with the problem before us or the essential dynamical content of the constitutive relation (1.3). The external force system required to effect the transformation must, in general, have a complex spatial structure whose length scale is comparable with that of the turbulence. Such an external force system must, for consistency, be regarded as changing the dynamical basis of the substructure of the ν-fluid from that represented by (1.4) and (1.5). Thus, in effect, the above comparison is between two *different* ν-fluids.

This criticism is not applicable in the case of *linear* material indifference, which restricts consideration to superimposition of non-rotating rigid-body motions. Here, the requisite external force field, though arbitrarily time-dependent, is exactly spatially uniform. It thus has a length scale which is infinitely large compared with that of the turbulence, and in no sense affects the dynamical substructure of the equivalent ν-fluid. Alternatively, by casting the constitutive relation (1.3) in terms only of the added stresses, we may note from (1.4) that all these linear superimposed rigid-body motions are dynamically possible without any external force field, since they may be produced through the effect of a superimposed uniform pressure gradient in the incompressible

Newtonian base fluid, leaving the Reynolds stresses invariant. From either approach we may conclude that a ν-fluid must satisfy the principle of linear material indifference. In its most general form, the constitutive relation (1.3) does not satisfy this principle; the necessary and sufficient condition for it to do so is that the fields

$$u, \frac{Du}{Dt}, \ldots, \frac{D^n u}{Dt^n}, \tag{1.9}$$

must not occur explicitly in the relation, though, of course, their spatial derivatives may do so.

As noted above, the principal of material indifference may not, in the case of turbulence, be extended to the rotational case, primarily because of the impossibility of balancing the Coriolis accelerations in the turbulence by an acceptable external force field. Hence, as Lumley (1970) has pointed out, no further invariance constraints should be placed on the constitutive relation (1.3). Indeed, if (1.3) were rotationally invariant, the decay of homogeneous stress in a stationary ν-fluid would be the same as that in the same ν-fluid rotating as a rigid body. This result is known not to be true in the equivalent cases of decaying homogeneous turbulence. This particular problem, and the way in which vorticity in a ν-fluid enters explicitly into the constitutive relation, is taken up in further detail in §3.2.

We now come to the important role played by the regularity condition on the function f in (1.3). In view of this condition, the function may be expanded in a tensor Taylor series about the origin of its argument space, the coefficients in the series being intrinsic properties of the ν-fluid. But it is a simple matter to see that only a finite number of these coefficients can have the dimensions of any power of ν. It follows that f must be a (dimensionally correct) polynomial in its arguments, so that the form of the constitutive relation for a ν-fluid is essentially determined by its order. For example, in view of the constraint (1.9), the most general constitutive relation for a first-order ν-fluid is

$$\dot{S} = p_1^1 \nu^{-1} S^2 + p_2^1 u' S + p_3^1 \nu S'' + p_4^1 \nu u'^2 + p_5^1 \nu \dot{u}' + p_6^1 \nu^2 u''', \tag{1.10}$$

where dots denote the total time derivatives, primes denote space derivatives, and the p's are non-dimensional constant isotropic tensors defining the fluid.

The restrictions which the regularity condition places on the constitutive relations, especially at high Reynolds numbers, are so great that the origin of the condition must clearly be examined. The background idea is that, for all non-zero values of ν, *every* possible set of initial conditions which are spatially well behaved shall lead to a solution which is unique and spatially well behaved for all subsequent times. I have not been able to show that the regularity condition follows in general from this idea; but nor have I been able to find a counter-example. If the result is generally true, it would seem that a proof must be concerned with the following argument.

In general, dimensional arguments would lead immediately to the conclusion that, for instance, the quantity p_1^1 in (1.10) may be any non-dimensional tensor function of non-dimensional scalar invariants of the velocity and stress fields.

The question then arises of how such non-dimensional scalar invariants can be constructed. For instance, one such function is

$$\left.\begin{aligned} p_1^1 &= p_1^1(\theta), \quad \text{where} \quad \theta = \theta_1/\theta_2, \\ \theta_1 &= \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\frac{\partial^3 u_p}{\partial x_q \partial x_r \partial x_s}\frac{\partial^3 u_p}{\partial x_q \partial x_r \partial x_s}, \\ \text{and} \qquad \theta_2 &= \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)^2. \end{aligned}\right\} \tag{1.11}$$

But it is then always possible to choose, as an initial condition for a flow problem, a velocity field which is everywhere regular and such that $\theta_1 = \theta_2 = 0$ at (say) $x_i = 0$, with the property that the limit of θ as $x_i \to 0$ is not independent of the path. Hence, if $\mathcal{S}$ is to be unique and spatially well behaved, p_1^1 may not depend upon θ. A similar argument would seem to be applicable to every possible non-dimensional scalar invariant like θ, with the result that $p_1^1 =$ constant. If this extension can be justified, then it would apply equally to all the defining tensors p_β^α in ν-fluids of all order, and the general regularity condition would be established.

We next distinguish between two essentially different types of ν-fluid: those in which

$$S_{ij} \equiv 0 \tag{1.12}$$

is a *possible* solution of the constitutive relation for arbitrary velocity distributions, and those in which this is not a possible solution. Henceforth, only fluids of the former type will be considered; the latter being more appropriate to problems involving the added stress system due to particle suspensions in a Newtonian fluid. The idea behind (1.12) is an extension of the notion of good behaviour of a dissipative system at lower Reynolds numbers. That (1.12) is true of the equations of turbulence is well known: laminar flow is always possible at all Reynolds numbers, and is actually necessary at sufficiently small Reynolds numbers. There is no suggestion here that the constitutive relation (1.3) can provide a quantitative model of turbulent motion at small Reynolds numbers, but it seems likely that the above features form an important structural property of the complete dynamical system which should be incorporated into the constitutive relation. This simplification is not necessary in order to make progress, but its consequences should be examined first. The conditions for (1.12) to be satisfied are very simple: in (1.10) for instance, we must have

$$p_4^1 = p_5^1 = p_6^1 = 0,$$

with analogous results for higher-order fluids.

Since the field equations for the dynamical substructure of a ν-fluid have a differential structure which is temporarily parabolic and spatially elliptic, it is to be expected that the field equations for a ν-fluid will have a similar structure, at least for finite Reynolds numbers. The mathematically natural specification for a ν-fluid flow will therefore be determined by initial conditions together with boundary conditions over a closed surface. Since the constitutive relation (1.3) is a differential equation for S of order n in time and $2n$ in space, the appropriate conditions would appear to be, in addition to the usual initial and boundary conditions on u, the specification within a closed surface Σ of the initial spatial distribution of S and its first $n-1$ time derivatives, together with n conditions on S and its first $2n-1$ normal space derivatives at each point of Σ.

The greatest interest naturally lies in the nature of the boundary conditions at an interface with a solid. This interest is not only concerned with the practical importance of such conditions, but it is connected with the foundations of the present approach which relies heavily on the principle of Reynolds number similarity in turbulent flow. The origin of this principle may be regarded as lying in the homogeneity of the no-slip condition,

$$u_i' = 0, \tag{1.13}$$

in the dynamical substructure of a ν-fluid. Thus, there is no forcing of the amplitude of the added-stress system at a solid boundary; it is only the geometrical configuration of the boundaries which influences the solution. If the principle of Reynolds number similarity is to survive, this situation must be reflected in the boundary conditions for a ν-fluid. This constraint is clearly consistent with the constraint (1.12).

The condition (1.13) immediately leads (with suitable assumptions about the spatial regularity of u_i') to the boundary conditions,

$$S \text{ has a double zero at a solid boundary.} \tag{1.14}$$

For a second-order ν-fluid, these boundary conditions are sufficient. It is therefore unfortunate that, as will shortly appear, ν-fluids of order less than three are seriously inadequate as models of turbulent flow. For it becomes necessary to postulate further boundary conditions. In principle, they should be derived from the field equations (1.4) and (1.5) for the substructure, just as, in principle, the constitutive relation (1.3) should be so derived. But the approach to the constitutive relation has been to rely on dimensional arguments and general notions of universality, and it would seem that, for consistency, the same approach to the boundary conditions should be adopted. In the case of a third-order ν-fluid, for instance, we are thus led to consider further boundary conditions of the form

$$a_1^3 u' S'' + a_2^3 \nu S^{\text{iv}} = 0, \tag{1.15}$$

where the a's are further non-dimensional isotropic tensor constants of the ν-fluid, which, in the case of turbulence, represent statistical properties of the Navier-Stokes equations.

The present paper does not in fact contain any examination of a boundary-value problem for a fully viscous third-order fluid, and it may well be that the above approach will need revision in the light of subsequent investigation. In the limit of infinite Reynolds number, the boundary-value problem has an entirely different structure (see §2.5).

We are now in a position to return to the central problem of constructing constitutive relations for turbulent flow at infinite Reynolds number. If, formally, the scales of u, S, length, and time, are all counted as $O(1)$, then the constitutive relations for low-order ν-fluids take the form

$$(n = 1), \quad \dot{S} = \nu^{-1} p_1^1 S^2 + O(\nu^0), \tag{1.16}$$

$$(n = 2), \quad \ddot{S} = \nu^{-2} p_1^2 S^3 + \nu^{-1}(p_2^2 S\dot{S} + p_3^2 u' S^2) + O(\nu^0), \tag{1.17}$$

$$(n = 3), \quad \dddot{S} = \nu^{-3} p_1^3 S^4 + \nu^{-2}(p_2^3 S^2 \dot{S} + p_3^3 u' S^3) + \nu^{-1}(p_4^3 S \ddot{S} + p_5^3 \dot{S}^2 + p_6^3 S^2 S'' + p_7^3 S S'^2 + p_8^3 u'^2 S^2) + O(\nu^0). \tag{1.18}$$

It is possible to deduce a few general properties of these dynamical systems, and compare them, in a purely phenomenological way, with those of turbulent motion at large Reynolds numbers.

In a first-order fluid, for instance, the limit equation becomes

$$p_1^1 S^2 = 0, \tag{1.19}$$

which, in view of the notation, is not a statement about the amplitude of the stress tensor, but about its relative geometry. Since the equations are universal, this can only be a condition of isotropy. Thus, at large Reynolds numbers, a first-order ν-fluid must in all cases have an approximately isotropic stress system. Significant departures from these conditions can persist only in layers of intense shear; elsewhere they must decay through transients of very short duration. While this behaviour has close affinities with that of the (negligible) contribution to the Reynolds stress from the smallest eddies of turbulent motion, it is quite unacceptable in a model which attempts to include the whole of the turbulent energy into the material properties of a ν-fluid.

The limit equation

$$p_1^2 S^3 = 0$$

for a second-order ν-fluid clearly has very similar properties to (1.9), and is unacceptable for the same reasons. If the fluid is degenerate, however, in the sense that

$$p_1^2 = 0, \tag{1.20}$$

the limit equation of (1.17) becomes

$$p_2^2 S \dot{S} + p_3^2 u' S^2 = 0, \tag{1.21}$$

with more interesting properties. In fact, the solutions of this equation would seem to be characteristic of the behaviour of a substantially larger range of (still small) eddy sizes than those represented by a first-order fluid. However, (1.21) is still too simple to include the whole of the turbulent energy, as can immediately be seen from the decay of homogeneous stress in a stationary fluid, or from the total absence of stress-transport terms.

Developing this sequence of ideas, we now see that the case of a doubly degenerate third-order ν-fluid is much more interesting. From (1.18) we have

$$p_1^3 = p_2^3 = p_3^3 = 0, \tag{1.22}$$

with the limit equation

$$p_4^3 S \ddot{S} + p_5^3 \dot{S}^2 + p_6^3 S^2 S'' + p_7^3 S S'^2 + p_8^3 u'^2 S^2 = 0. \tag{1.23}$$

Perhaps the most important single property of (1.23) is the existence of a bounded relaxation time for the stress in the limit $\nu \to 0$. The lowest order ν-fluid which can achieve this is a third-order one, and it must be doubly degenerate. A similar remark applies to the length-scale associated with the transport terms in $S^2 S''$

and SS'^2. Further, the equation includes the possibility of a steady homogeneous stress system in the presence of uniform strain, the equilibrium being given by

$$p_8^3 u'^2 S^2 = 0. \tag{1.24}$$

Since (1.24) is homogeneous in both u' and S, the solution will obviously be such that the stress-type is a function only of the strain-type, the amplitude of neither being relevant. For fluid constants which give real solutions to (1.24), the structure of the dynamical equation (1.23) may then be described as the response of the decay and transport terms to departures from equilibrium. Finally, we may note the possibility of a stress-entrainment process essentially governed by a non-linear wave equation of the type

$$S\ddot{S} - S^2 S'' = 0. \tag{1.25}$$

All of the points mentioned in the preceding paragraph find an important place in the theory of turbulence. The ν-fluid approach may not be able to provide a quantitative model of turbulence, but a relatively simple self-consistent dynamical system with such strong affinities to turbulence would seem to deserve further study in its own right. The remainder of the paper is concerned with some aspects of such a study.

2. The mechanics of a doubly-degenerate third-order ν-fluid at infinite Reynolds number

2.1. *The decay of homogeneous stress in a fluid at rest*

When there is no flow and no spatial variation of stress, the equation of motion is satisfied identically, and the constitutive relation (1.23) becomes

$$p_4^3 S\ddot{S} + p_5^3 \dot{S}^2 = 0. \tag{2.1}$$

Since S is a symmetric tensor, and (2.1) is the limiting form of an expression for a time-derivative of S, the most general possible form of (2.1) is

$$\begin{aligned} a_1 S_{kk}\ddot{S}_{ij} + a_2 S_{ij}\ddot{S}_{kk} + a_3(S_{ik}\ddot{S}_{jk} + S_{jk}\ddot{S}_{ik}) + a_4 S_{kk}\ddot{S}_{kk}\delta_{ij} + a_5 S_{kl}\ddot{S}_{kl}\delta_{ij} \\ + a_6 \dot{S}_{kk}\dot{S}_{ij} + a_7 \dot{S}_{ik}\dot{S}_{jk} + a_8 \dot{S}_{kk}^2\delta_{ij} + a_9 \dot{S}_{kl}\dot{S}_{kl}\delta_{ij} = 0, \end{aligned} \tag{2.2}$$

where the a's, being invariants of the tensors p_4^3 and p_5^3 are to be regarded as being intrinsic constants of the fluid.

The simplest case is obviously that of an isotropic stress system, for which (2.2) reduces to

$$S_{kk}\ddot{S}_{kk} - \frac{2+\beta}{1+\beta}\dot{S}_{kk}^2 = 0, \tag{2.3}$$

where β is a constant fixed by the a's. The general solution is then

$$S_{kk} = \frac{R}{(t-t_0)^{1+\beta}}, \tag{2.4}$$

where the constants R and t_0 are fixed by the initial values of S_{kk} and $\dot{S}_{kk}$. For small values of β, this result is compatible with the observations of Batchelor & Townsend (1948) on decaying, approximately isotropic turbulence. An examination of the effect of finite Reynolds number (§3) suggests that β must be positive;

for the present limiting analysis, however, little is to be gained from considering non-zero values of β.

While a full examination of (2.2) in the anisotropic case would doubtless be of theoretical interest, the limited extent of experimental evidence on the decay of homogeneous turbulence would make such an investigation scarcely worthwhile as an attempt to evaluate the fluid constants. It is sufficient to enquire whether the constitutive relation is compatible with the two principal processes of decay at large Reynolds numbers, namely, the decay of internal energy, and a general tendency towards stress-isotropy. A single example can answer this question.

Consider, then, the constitutive relation

$$S_{kk}\ddot{S}_{ij} - 2\dot{S}_{kk}\dot{S}_{ij} + \tfrac{1}{2}(1+n)(1+m)(S_{ij} - \tfrac{1}{3}S_{kk}\delta_{ij})\ddot{S}_{kk} \\ - (1+n+m)\dot{S}_{kk}(\dot{S}_{ij} - \tfrac{1}{3}\dot{S}_{kk}\delta_{ij}) = 0, \quad (2.5)$$

where n and m are positive constants related to the a's of (2.2). The general solution of this system of differential equations is

$$S_{ij} = \frac{R}{t-t_0}\delta_{ij} + \frac{A_{ij}}{(t-t_0)^{1+n}} + \frac{B_{ij}}{(t-t_0)^{1+m}}, \quad (2.6)$$

where R and t_0 are scalar constants, and A_{ij} and B_{ij} are symmetric tensor constants subject to the condition

$$A_{kk} = B_{kk} = 0. \quad (2.7)$$

Thus, (2.6) involves twelve arbitrary constants of integration, to be fixed by the six initial conditions on each of S_{ij} and $\dot{S}_{ij}$. Provided both n and m are positive, the solution (2.6) represents a general tendency towards isotropy, the relaxation time compared with the relaxation time for energy decay being adjustable through choice of n and m. Moreover, the whole solution appears to be thermodynamically acceptable for arbitrary initial conditions, subject only to the constraints

$$\dot{S}_{kk}(0) < 0 \quad \text{and} \quad \lambda_i\lambda_j S_{ij}(0) > 0 \quad \text{for all } \lambda_i.$$

2.2. *The effect of rotation on the decay of homogeneous stress*

Another dynamical process which might usefully be illustrated by a simple example is the effect of a rigid-body rotation on the decay of homogeneous stress. This process, which is intimately related to the invalidity of the principle of rotational material indifference, has already been referred to in § 1, and the present purpose is merely to illustrate the general nature of results which might be expected.

For this purpose we consider a particularly simple ν-fluid in which the last term of the constitutive relation (1.23) takes, in the absence of a rate of pure strain, the form

$$p_8^3 u'^2 S^2 \equiv \kappa^2\mu^2\omega_{il}\omega_{jl}S_{kk}^2, \quad (2.8)$$

where $\kappa\mu$ is a non-dimensional constant, and

$$\omega_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}. \quad (2.9)$$

A more critical evaluation of this form is taken up in §2.4; for the moment we note only that it has the required symmetry, and is such that there is no equilibrium solution other than $S_{kk} = 0$ when the motion is a rigid-body rotation.

Combination of (2.5) and (2.8) gives

$$S_{kk}\ddot{S}_{kk} - 2\dot{S}_{kk}^2 + 2\kappa^2\mu^2\omega^2 S_{kk}^2 = 0, \tag{2.10}$$

where ω is the constant angular velocity, with general solution

$$S_{kk} = 2\kappa\mu R\omega \frac{\exp\{-\kappa\mu\omega(t-t_0)\}}{1-\exp\{-2\kappa\mu\omega(t-t_0)\}}, \tag{2.11}$$

in which R and t_0 are again the constants of integration. When $\omega(t-t_0)$ is small, (2.11) reduces to

$$S_{kk} \sim \frac{R}{t-t_0},$$

in agreement with (2.4). When $\omega(t-t_0)$ is large, the asymptotic decay is given by

$$S_{kk} \sim 2\kappa\mu R\omega \exp\{-\kappa\mu\omega(t-t_0)\}. \tag{2.12}$$

Thus, when the relaxation time of the natural decay process is short compared with the reciprocal of the rotation rate, the dynamics of the substructure of the ν-fluid is essentially unaffected by rotation (this being also the normal situation in the usual continuum approximation to a molecular gas). When this inequality is not satisfied, the rotation has a substantial effect and acts, as might be expected, as a powerful suppressor of stress.

The effect of the rotation on the relative geometry of the stress tensor is more complicated. In the case of isotropic initial conditions, however, the stress tensor must remain axially symmetric, and it becomes a simple matter to examine the situation during the asymptotic régime (2.12). Taking the rotation vector to be parallel to the x_3-axis, we find that the (3, 3) component of the constitutive relation becomes (using (2.12))

$$\begin{aligned}\ddot{S}_{33} + \kappa\mu\omega(3+n+m)\,\dot{S}_{33} + \tfrac{1}{2}\kappa^2\mu^2\omega^2(1+n)\,(1+m)S_{33} \\ = \tfrac{1}{3}\kappa^2\mu^2\omega^2[\tfrac{1}{2}(1+n)\,(1+m)-(1+n+m)]\,S_{kk}.\end{aligned} \tag{2.13}$$

The transient solutions of (2.13) must clearly by asymptotically negligible compared with the forced solution, for, otherwise, the internal energy associated with the S_{33} component of stress would be greater than the total available. Similarly, the forced solution itself must satisfy the constraint $S_{33} \leqslant S_{kk}$. These constraints arise from the origin of stress as momentum transport by a real velocity field and demand, in the particular fluid under consideration, a fairly powerful process of tendency to isotropy represented by the condition

$$nm-(n+m) \geqslant 4. \tag{2.14}$$

The extreme case of equality in (2.14) gives rise to an asymptotic one-dimensional stress system $S_{33} = S_{kk}$. For progressively larger values of the left-hand side of (2.14), the asymptotic stress system becomes progressively more isotropic, with $S_{33} = kS_{kk}$ and

$$k = \frac{1}{3}\frac{nm-(n+m)-1}{nm-(n+m)-3}. \tag{2.15}$$

2.3. *Equilibrium in irrotational rate of strain*

We now turn to a central dynamical problem in the motion of ν-fluids. This concerns the nature of the spatially homogeneous stress tensor which can be in equilibrium with a spatially homogeneous field of irrotational rate of strain. Ideas about the nature of such an equilibrium have been made the basis of a co-ordinated view of turbulent phenomena by Townsend (1956) and later authors. To this extent, the present paper may be regarded as providing an elementary mathematical model of such ideas, together with models of the attendant spatial and temporal processes which are brought into play when the equilibrium is disturbed.

In this problem, we are concerned solely with the term,

$$p_8^3 u'^2 S^2, \tag{2.16}$$

in the constitutive relation for a doubly degenerate third-order ν-fluid, and it is unfortunate that the complexity of its geometrical structure should be so great. There are no less than twenty independent scalar invariants of the tenth-order tensor p_8^3 which survive after all the constrains of symmetry and incompressibility, including the irrotationality of the rate-of-strain field, have been imposed. (For the sake of completeness, these twenty terms are set out explicitly in the appendix.) Nevertheless, it is a simple matter to see that, in general, a third-order ν-fluid cannot have a homogeneous distribution of stress in the presence of an irrotational rate of strain, other than $S_{kk} = 0$. The condition for a non-trivial solution to exist is a set of relations between the scalar invariants of p_8^3 which define the fluid, and these relations divide ν-fluids into two distinct classes whose dynamical behaviour at large Reynolds numbers is radically different. No further attention is given in the present paper to fluids in which equilibrium is impossible.

As noted in §1, the homogeneity of (2.16) in both u' and S makes impossible a unique solution to the equilibrium equation. We may therefore normalize the stress and rate-of-strain tensors, without loss of generality. Thus, we set

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad e_{kk} = 0, \tag{2.17}$$

$$S_{kk} = e_{kl} e_{kl} = 1, \tag{2.18}$$

and adopt the corresponding non-dimensional normalized meanings of S_{ij} and e_{ij} throughout the remainder of §2.3. There is now only one remaining scalar invariant of e_{ij}, namely

$$\phi = e_{kl} e_{lm} e_{mk} = 3 e_1 e_2 e_3, \tag{2.19}$$

where e_1, e_2, e_3, are the principal rates of strain. Clearly ϕ defines the rate-of-strain type, being zero for a plane strain and $\pm 1/\sqrt{6}$ for axially symmetric extension and compression, respectively.

If the equilibrium equation is such that S_{ij} (with its new meaning) is uniquely determined by ϕ, simple invariant analysis shows that S_{ij} must have the structure

$$S_{ij} = \tfrac{1}{3}\{1 - g(\phi)\}\delta_{ij} + f(\phi)\, e_{ij} + g(\phi)\, e_{ik} e_{jk}, \tag{2.20}$$

where $f(\phi)$ and $g(\phi)$ are non-dimensional scalar functions of ϕ. Further, the equilibrium term (2.16), being a second-order symmetric tensor function of e_{ij} must also have this form:

$$p_8^3 u'^2 S^2 \equiv A\delta_{ij} + Be_{ij} + Ce_{ik}e_{jk}, \tag{2.21}$$

where A, B, C, depend on $\phi, f(\phi), g(\phi)$, and the fluid constants associated with p_8^3. Thus, the necessary and sufficient conditions for equilibrium are given by the three equations

$$A = B = C = 0. \tag{2.22}$$

These three constraints on the functions f and g illustrate the mathematical nature of the dynamical problem as one of algebraic compatability.

In more detail, the equations (2.22) take the form

$$\left.\begin{aligned} Q_1 + \phi Q_2 + \phi^2 Q_3 &= 0, \\ Q_4 + \phi Q_5 &= 0, \\ Q_6 + \phi Q_7 &= 0, \end{aligned}\right\} \tag{2.23}$$

where the Q_i are quadratic functions of f and g (and otherwise independent of ϕ), with constant coefficients which are linear combinations of the invariants of p_8^3. (The details of these quadratic functions are also set out in the appendix.) For each value of ϕ, each of the equations (2.23) is that of a conic in (f, g)-space, and the equilibrium condition may be regarded as the condition that these three conics be concurrent for all values of ϕ.

For general fluid constants, the equations (2.23) are incompatible, and there is no solution. In a proper sense, the next most general situation is that in which there is a solution with

$$f(\phi) = \text{constant}, \quad g(\phi) = \text{constant}. \tag{2.24}$$

For, a necessary condition for a solution point (f, g) is that it should lie on both ϕ-eliminants of the system (2.23), namely, the quartic curve

$$Q_4 Q_7 - Q_5 Q_6 = 0, \tag{2.25}$$

and the sextic curve

$$Q_2 Q_4 Q_5 - Q_3 Q_4^2 - Q_1 Q_5^2 = 0. \tag{2.26}$$

In general, these two curves will have only a finite number of common points, so that f and g cannot be functions of ϕ. The sufficient condition for a solution is then that one of these common points shall lie on one of the conics (2.23) for all values of ϕ, that is, shall lie at the intersection of (say) Q_4 and Q_5. In order for f and g to be functions of ϕ, (2.25) and (2.26) must clearly have a continuous curve in common, which requires more stringent conditions on the fluid constants.

A full examination of this problem of algebraic consistency, and the restriction which it places on the possible forms of the functions $f(\phi)$ and $g(\phi)$, would clearly be of theoretical interest. Once again, however, the limited experimental evidence scarcely justifies such an examination, particularly as this evidence appears to be compatible with a specially simple solution belonging to the class (2.24), namely,

$$f(\phi) = \text{constant} = -\lambda, \; g(\phi) \equiv 0. \tag{2.27}$$

Such a solution gives, in the limited case of irrotational equilibrium only, the stress tensor a virtual viscosity structure

$$S_{ij} = \tfrac{1}{3}\delta_{ij} - \lambda e_{ij}. \tag{2.28}$$

For $\lambda = 0.20$, (2.28) gives table 1, which shows values close to those suggested by Townsend (1956).

	S_{11}	S_{22}	S_{33}
Plane strain	0·19	0·33	0·48
Axially symmetric extension	0·16	0·42	0·42
Axially symmetric compression	0·25	0·25	0·50

TABLE 1

The conditions on the fluid constant p_8^3 which permit the existence of the solution (2.27) are such that there are five linear relations between the twenty defining constants of p_8^3. There is thus a fifteen-parameter family of ν-fluids with the same equilibrium solution (2.27). These undetermined constants are mainly concerned with the dynamical response of the decay and transport terms in the constitutive relation to departures from equilibrium. However, they are also partly concerned with the uniqueness of the equilibrium solution itself; a matter which can most easily be discussed in terms of an example.

Perhaps the simplest ν-fluid which has the required equilibrium structure is that in which (2.16) takes the form (in the absence of vorticity)

$$-\kappa^2\{(S_{il} - \tfrac{1}{3}S_{kk}\delta_{il})(S_{jl} - \tfrac{1}{3}S_{kk}\delta_{jl})\, e_{mn}e_{mn} - \lambda^2 S_{kk}^2 e_{il}e_{jl}\} = 0, \tag{2.29}$$

which clearly has the solution (2.28). Moreover, when none of the principal rates of strain is zero, it is a straightforward matter to show that the solution (2.28) is unique, thus rendering the structural assumption (2.20) self-consistent. In the case of a plane rate of strain, however, the solution is not unique. Taking the x_3-axis to be that along which there is no extension, we find that the equations for S_{33} and S_{12} (with respect to principal axes of e_{ij}) are

$$(S_{33} - \tfrac{1}{3}S_{kk})^2 = 0, \tag{2.30}$$

and

$$S_{12}(S_{11} + S_{22} - \tfrac{2}{3}S_{kk}) = 0. \tag{2.31}$$

The unique solution of (2.30) thus makes the bracket in (2.31) zero, leaving S_{12} arbitrary. The equations for the remaining normal stresses, namely,

$$\left.\begin{aligned}(S_{11} - \tfrac{1}{3}S_{kk})^2 + S_{12}^2 &= \tfrac{1}{2}\lambda^2 S_{kk}^2,\\ (S_{22} - \tfrac{1}{3}S_{kk})^2 + S_{12}^2 &= \tfrac{1}{2}\lambda^2 S_{kk}^2,\end{aligned}\right\} \tag{2.32}$$

then have solutions for all S_{12}, subject to the usual condition that the principal stresses shall be positive. In such a fluid, therefore, the equilibrium conditions determine the principal stresses in terms of the fluid constants,

$$S_{11}^* = (\tfrac{1}{3} - 2^{-\frac{1}{2}}\lambda)S_{kk}, \quad S_{22}^* = (\tfrac{1}{3} - 2^{-\frac{1}{2}}\lambda)S_{kk}, \quad S_{33}^* = \tfrac{1}{3}S_{kk}, \tag{2.33}$$

but not their orientation relative to the principal axes of rate-of-strain, the latter being fixed, in the case of strict spatial homogeneity, by the history of the flow in a more completely specified initial-value problem.

Since it is possible to choose the fluid constants in such a way that the above indeterminacy in equilibrium is removed, giving the constitutive relation a structure different from (2.29), this distinction becomes a criterion for dividing ν-fluids, once again, into two different classes. The dynamical consequences of the distinction are substantial, a point which is taken up further after considering the effect of rotation.

2.4. *Equilibrium in the presence of rotation*

We now return to the example (2.8) of a ν-fluid in which the constitutive relation is very simply affected by vorticity. The equation of equilibrium for such a fluid is

$$-\kappa^2\{(S_{il}-\tfrac{1}{3}S_{kk}\delta_{il})(S_{jl}-\tfrac{1}{3}S_{kk}\delta_{jl})e_{mn}e_{mn}-\lambda^2S_{kk}^2e_{il}e_{jl}-\mu^2S_{kk}^2\omega_{il}\omega_{jl}\}=0. \quad (2.34)$$

Moreover, the equation of motion (1.7) requires that, for homogeneous stress distributions, the rate of pure strain shall be plane, with the rotation vector aligned along the direction of the zero rate of strain. Hence, we need only consider the tensors

$$\left.\begin{aligned} e_{11} &= -e_{22} = e, \\ \omega_{12} &= -\omega_{21} = \omega, \end{aligned}\right\} \quad (2.35)$$

with every other component of both tensors zero.

In the axes defined by (2.35), the equilibrium equations for S_{33} and S_{12} reduce to (2.30) and (2.31), with the same resulting arbitrariness in S_{12}. The remaining normal stresses are then given by

$$\left.\begin{aligned} (S_{11}-\tfrac{1}{3}S_{kk})^2+S_{12}^2 &= (\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\mu^2\omega^2/e^2)S_{kk}^2, \\ (S_{22}-\tfrac{1}{3}S_{kk})^2+S_{12}^2 &= (\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\mu^2\omega^2/e^2)S_{kk}^2, \end{aligned}\right\} \quad (2.36)$$

which again fix only the magnitude of the principal stresses,

$$\left.\begin{aligned} S_{11}^* &= \{\tfrac{1}{3}-(\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\mu^2\omega^2/e^2)^{\frac{1}{2}}\}S_{kk}, \\ S_{22}^* &= \{\tfrac{1}{3}+(\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\mu^2\omega^2/e^2)^{\frac{1}{2}}\}S_{kk}, \end{aligned}\right\} \quad (2.37)$$

their orientation relative to the principal axes of rate of strain remaining undetermined by the equilibrium conditions. Whatever the status of this arbitrariness in the equilibrium solution may be in the irrotational case, it seems that, when vorticity is present, the arbitrariness has dynamical consequences which are unacceptable, and that the vorticity must determine the orientation of the principal axes of stress relative to those of the rate of strain. The point at issue concerns steady uniform shear, and it is necessary to anticipate results of the following section. It appears that for these ν-fluids which, like turbulence, exhibit a universal logarithmic law for the velocity distribution near a plane boundary in such flows, the stress tensor must, to the first order, be in equilibrium at the boundary, and the equilibrium conditions must be such that the entire stress tensor is determined uniquely, apart from its scale S_{kk}. The equilibrium constitutive relation (2.34) thus requires modification.

A simple modification of (2.34) which has the required property is

$$\kappa^2\{(S_{il}-\tfrac{1}{3}S_{kk}\delta_{il})(S_{jl}-\tfrac{1}{3}S_{kk}\delta_{jl})e_{mn}e_{mn}-\lambda^2S_{kk}^2e_{il}e_{jl}-\mu^2S_{kk}^2\omega_{il}\omega_{jl}$$
$$+k(S_{ij}-\tfrac{1}{3}S_{kk}\delta_{ij})(S_{kl}e_{km}\omega_{lm}-\alpha S_{kk}\omega_{lm}\omega_{lm})\}=0. \quad (2.38)$$

The equilibrium equations for S_{33} and S_{12} in this fluid are (in the principal axes of e_{ij})

$$(S_{33}-\tfrac{1}{3}S_{kk})\{e^2(S_{33}-\tfrac{1}{3}S_{kk})-k\omega(eS_{12}+\alpha\omega S_{kk})\}=0, \quad (2.39)$$

$$S_{12}\{-e^2(S_{33}-\tfrac{1}{3}S_{kk})-k\omega(eS_{12}+\alpha\omega S_{kk})\}=0, \quad (2.40)$$

of which the unique solution with non-zero S_{12} is

$$S_{12}=-\alpha\omega/e, \quad (2.41)$$

$$S_{33}=\tfrac{1}{3}S_{kk}. \quad (2.42)$$

The remaining principal stresses continue to be given by (2.37).

When $\omega=e$, the flow is a uniform plane shear. In axes along (x_1) and orthogonal (x_2, x_3) to the flow, the stresses are then

$$\left.\begin{aligned} S_{11}&=(\tfrac{1}{3}+\alpha)S_{kk}=0{\cdot}60S_{kk},\\ S_{22}&=(\tfrac{1}{3}-\alpha)S_{kk}=0{\cdot}07S_{kk},\\ S_{33}&=\tfrac{1}{3}S_{kk}=0{\cdot}33S_{kk},\\ S_{12}&=-(\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\mu^2-\alpha^2)^{\frac{1}{2}}S_{kk}=-0{\cdot}09S_{kk},\end{aligned}\right\} \quad (2.43)$$

where the numerical values corresponding to $\mu=0{\cdot}33$ and $\alpha=0{\cdot}26$ have been chosen to provide reasonable agreement with observations in the constant-stress layer of a turbulent flow.

It is interesting that this estimate of μ should be such that, for values of ω/e only slightly in excess of unity, equilibrium becomes impossible without the support of (unacceptable) tensions in the added stress system. Thus, in these closed-streamline flows of the ν-fluid, the only true equilibrium solution is $S_{kk}=0$, though the possibility of a pulsating quasi-equilibrium cannot be ruled out.

2.5. *Two-dimensional flow near a plane boundary*

Consider, now, the steady uni-directional flow of a ν-fluid along a two-dimensional channel under the influences of a constant pressure gradient. From (1.23), the constitutive relation becomes

$$p_6^3S^2S''+p_7^3SS'^2+p_8^3u'^2S^2=0. \quad (2.44)$$

If this equation is to represent, in a qualitatively acceptable way, the turbulent flow of a Newtonian fluid in the central (inviscid) region of such a channel, the velocity gradient u' must have a simple-pole singularity on the boundaries. The basis of this well-confirmed 'law of the wall' is very general, and follows from little more than dimensional arguments and an assumption of overlapping régimes of viscous and inviscid flow in a vanishingly thin layer of constant stress at the boundaries. The result should therefore be true for a whole class of ν-fluids and, in effect, becomes a boundary condition for the limit equation (2.44).

If the stress at the boundary is to be bounded, there are, locally, two ways in which this boundary condition can be satisfied. In the first, S is regular at the boundary, so that the first two terms in (2.44) are bounded at the boundary. Hence, the last term must be bounded at the boundary, which implies, in view of the singularity in u'^2, that S must be a double solution of the equilibrium equation $p_8^3 u'^2 S^2 = 0$ at the boundary. But such boundary conditions always lead to an over-determination of the inviscid flow problem. For, in the flow under consideration, elimination of u' from the four† equations (2.44) yields three second-order differential equations for the three normal stresses, the distribution of shear stress being known from the equation of motion. Hence, if the constitutive relation determines the equilibrium structure of S uniquely, there will be six boundary conditions on the normal stresses at each boundary arising from the double solution of the equilibrium equation. Clearly there is, in general, no solution to such a differential system. Moreover, even when the constitutive relation is such that only the principal stresses, not their orientation, are determined by the equilibrium equations, there are still too many boundary conditions (four) at each boundary to permit a solution in general.

That there should be no solution under the above assumptions concerning the regularity of S is encouraging, since such a solution would be incompatible with some of the ideas underlying the 'law of the wall'. In particular, the regularity would imply that the non-dimensional constants in that law would be determined by the solution of a boundary-value problem and could not therefore be universal. Alternatively, the regularity is incompatible with the idea that the only relevant length-scale in the local flow is the distance from the boundary.

The above difficulties are all resolved in the second way of satisfying (2.44) near a boundary, which requires that S shall have a singularity at the boundary of a type which leaves S bounded: typically,

$$S \sim S_0 + S_1 y^n \quad (0 < n < 1), \tag{2.45}$$

where y is the distance from the boundary. The boundedness of S_0 again requires it to have an equilibrium structure, but in this case it need only be a single solution of the equilibrium equations, since the transport and equilibrium terms in (2.42) both diverge at the boundary in a manner determined by n, which itself is determined as a universal constant in terms of the fluid constants p_6^3 and p_8^3. The sole boundary condition is therefore

$$S \text{ as an equilibrium structure at the boundary.} \tag{2.46}$$

The dynamical consequence of the absence of arbitrariness in the equilibrium solutions is now apparent. Since the shear stress is known, (2.44) determines the three normal stresses at each boundary, and, apart from difficulties associated with the application of boundary conditions at singular points of the differential equations, these conditions are just sufficient for the sixth-order differential system for the normal stresses. Thus, it is reasonable to suppose that a ν-fluid with a unique equilibrium structure, and whose constants are such that n satisfies

† By symmetry, the equations in two of the shear stresses are identically zero.

the inequality (2.35), will, with the boundary condition (2.46) give rise to a universal 'law of the wall' of the required form.

Further progress is again made difficult by the very complicated structure of the general form of the transport terms in (2.44) and the limited object of the remainder of this section is merely to illustrate the nature of the dynamical problem by the motion of a particularly simple ν-fluid whose transport terms are chosen on a basis of little more than analytic convenience. Such a choice is

$$bS_{mn}\frac{\partial^2}{\partial x_m \partial x_n}\{(S_{il}-\tfrac{1}{3}S_{kk}\delta_{il})(S_{jl}-\tfrac{1}{3}S_{kk}\delta_{jl})\}, \tag{2.47}$$

where b is a further non-dimensional constant for the fluid. The only general feature of (2.47) that is worth noting is the representation of the transport of a quadratic function of the stress by an elliptic differential operator. In other respects the choice is without significance, and can scarcely be expected to provide a complete model of turbulent transport processes.

For two-dimensional flow in the direction of x_1 in the layer of constant shear stress τ_0 near the boundary $x_2 = 0$, the constitutive relations become

$$\{(S_{11}-\tfrac{1}{3}S_{kk})^2\}'' = F\{(S_{11}-\tfrac{1}{3}S_{kk})^2+\tau_0^2-\tfrac{1}{2}(\lambda^2+\mu^2)S_{kk}^2 + k(S_{11}-\tfrac{1}{3}S_{kk})(\tfrac{1}{2}S_{11}-\tfrac{1}{2}S_{22}-\alpha S_{kk})\}, \tag{2.48}$$

$$\{(S_{22}-\tfrac{1}{3}S_{kk})^2\}'' = F\{(S_{22}-\tfrac{1}{3}S_{kk})^2+\tau_0^2-\tfrac{1}{2}(\lambda^2+\mu^2)S_{kk}^2 + k(S_{22}-\tfrac{1}{3}S_{kk})(\tfrac{1}{2}S_{11}-\tfrac{1}{2}S_{22}-\alpha S_{kk})\}, \tag{2.49}$$

$$\{(S_{33}-\tfrac{1}{3}S_{kk})^2\}'' = F\{(S_{33}-\tfrac{1}{3}S_{kk})^2+k(S_{33}-\tfrac{1}{3}S_{kk})(\tfrac{1}{2}S_{11}-\tfrac{1}{2}S_{22}-\alpha S_{kk})\}, \tag{2.50}$$

and

$$(S_{33}-\tfrac{1}{3}S_{kk})'' = F\{(S_{33}-\tfrac{1}{3}S_{kk})-k(\tfrac{1}{2}S_{11}-\tfrac{1}{2}S_{22}-\alpha S_{kk})\}, \tag{2.51}$$

where primes denote differentiation with respect to x_2, and

$$F = 2\kappa u'^2/bS_{22}. \tag{2.52}$$

The interest lies in finding the solution of this system of equations in the neighbourhood of a plane of equilibrium at which, from (2.43),

$$\left.\begin{aligned} S_{11} &= (\tfrac{1}{3}+\alpha)S_0, \\ S_{22} &= (\tfrac{1}{3}-\alpha)S_0, \\ S_{33} &= \tfrac{1}{3}S_0, \\ S_0 &= (\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\mu^2-\alpha^2)^{-\frac{1}{2}}\tau_0. \end{aligned}\right\} \tag{2.53}$$

It appears that, for this fluid, the expected singularity is associated only with the overall scale S_{kk}, so that we set

$$\left.\begin{aligned} S_{11} &= (\tfrac{1}{3}+\alpha)S_{kk}+T_1, \\ S_{22} &= (\tfrac{1}{3}-\alpha)S_{kk}+T_2, \\ S_{33} &= \tfrac{1}{3}S_{kk}-(T_1+T_2), \end{aligned}\right\} \tag{2.54}$$

where T_1, T_2, are regular at $x_2 = 0$, with $T_1(0) = T_2(0) = 0$, and S_{kk} is singular at $x_2 = 0$ in such a way that $S_{kk}(0) = S_0$, and $S'_{kk}(0)$ is unbounded. Equations (2.50) and (2.51) then give

$$T'_1(0) = \frac{k-2}{k+2}\, T'_2(0), \tag{2.55}$$

and

$$F \sim 1/x_2^2 \quad \text{as} \quad x_2 \to 0. \tag{2.56}$$

The remaining equations reduce, for small values of x_2, to

$$(S_{kk}^2 - S_0^2)'' = -\frac{\tau_0^2}{\alpha^2 S_0^2 x_2^2}\,(S_{kk}^2 - S_0^2), \tag{2.57}$$

with general solution

$$S_{kk}^2 - S_0^2 = S_1^2 x_2^{n_1} + S_2^2 x_2^{n_2}, \tag{2.58}$$

where S_1 and S_2 are arbitrary constants, and n_1 and n_2 are the roots of

$$n(n-1) = -\tau_0^2/\alpha^2 S_0^2. \tag{2.59}$$

Since the right-hand side of (2.59) is negative, both n_1 and n_2 lie in the interval $(0, 1)$ as required by the constraint (2.45) on the singularity. The solution is therefore self-consistent.

The result (2.56) gives with (2.52),

$$u \sim \frac{\tau_0^{\frac{1}{2}}}{K} \log \frac{x_2}{l}, \tag{2.60}$$

where l in an arbitrary length, and

$$K = \left[\frac{2\kappa(\frac{1}{2}\lambda^2 + \frac{1}{2}\mu^2 - \alpha^2)^{\frac{1}{2}}}{b(\frac{1}{3} - \alpha)}\right]^{\frac{1}{2}}. \tag{2.61}$$

The constant K is thus expressible entirely in terms of the fluid constants, so that the logarithmic law (2.60) takes a universal form. In this respect, the illustration is typical of a wide class of ν-fluids. Significant comparison of these results with observations on turbulence, however, would be out of place without a much more exhaustive examination of both the effect of vorticity on the equilibrium solutions, and the possible structure of the transport terms in the constitutive relation.

The above sketch of the potential properties of third-order ν-fluids at infinite Reynolds number has been largely based on particular cases which illustrate, in the simplest possible manner, dynamical processes of a kind which occur in turbulent motion. The results for these cases are encouraging, and suggest that a further study of the motion of ν-fluids at infinite Reynolds number may be rewarding. The present paper does not analyse further any of these problems. In §3, we return to the question, raised in §1, of the nature of the singular perturbation generated by large but finite Reynolds numbers. The only dynamical problem considered in this context is the very special one of decay of isotropic stress, largely because the geometrical simplicity of this problem enables all doubly degenerate third-order ν-fluids to be considered with complete generality.

3. The effect of finite Reynolds number on the decay of isotropic stress

For a doubly degenerate third-order ν-fluid, which is at rest and has a homogeneous and isotropic stress system, the constitutive relation becomes

$$\ddot{S}_{kk} = \gamma\nu^{-1}(S_{kk}\ddot{S}_{kk} - \beta\dot{S}^2_{kk}), \tag{3.1}$$

where β and γ are constants of the fluid. The transformation

$$S_{kk} = S_0 f(x), \quad t = \nu x/\gamma S_0, \tag{3.2}$$

reduces this relation to the one-parameter family of differential equations

$$f''' = ff'' - \beta f'^2. \tag{3.3}$$

It is obviously an intrinsic property of any third-order ν-fluid that the decay of isotropic stress should be governed by only three parameters, namely, the initial conditions for (3.3). But this circumstance raises problems of self-consistency. For, in the problem under consideration, there is no supply of energy to the stress system, so that for a continuous range of values of $f''(0)$ and any pair of values $f(0) > 0$ and $f'(0) < 0$, the resulting decay law must be such that, for all $x > 0$, (i) $f(x) > 0$ and $f'(x) < 0$ and (ii) $f(x) \to 0$ as $x \to \infty$. One might have expected these energetic constraints to impose conditions on β; but in fact they do not.

The transformation

$$f' = -f^2\phi(g), \quad g = \log f, \quad \phi' = \frac{d\phi}{dg}, \tag{3.4}$$

reduces the Falkner–Skan equation (3.3) to the first-order differential equation

$$\phi\phi'\frac{d\phi'}{d\phi} = -7\phi\phi' - \phi'^2 - 6\phi^2 + (2-\beta)\phi + \phi', \tag{3.5}$$

whose solutions in the phase-plane may be examined in the usual manner.

The value $\beta = 2$ is clearly critical. For this value, the differential equation has two singular points: a node at the origin; and a saddle point at $\phi = 0$, $\phi' = 1$. The two solutions through the saddle point, one of which is $\phi \equiv 0$, divide the plane into four regions, with typical solutions in each region as sketched in figure 1. The arrows on these solutions indicate the direction of time for a decaying stress-system, with the right-hand and left-hand halves of the figure corresponding to $\gamma > 0$ and $\gamma < 0$, respectively.

The relevant asymptotic behaviours are:

$$\phi \to 0, \quad \phi' \to 0, \quad \phi' \sim 6\phi^2; \tag{3.6}$$

$$\phi \to 0, \quad \phi' \to \pm\infty, \quad \phi' \sim \text{constant}/\phi; \tag{3.7}$$

$$\phi \to -\infty, \quad \phi' \to +\infty, \quad \phi' \sim -2\phi. \tag{3.8}$$

Examination of these behaviours shows that regions (2), (3), and (4) must be rejected. In region (3), every solution is such that S_{kk} changes sign through zero after a finite time; and in regions (2) and (4), every solution is such that S_{kk} passes through a minimum and subsequently increases after a finite time.

As $t \to \infty$ in region (1), however, every solution asymptotes to the universal form (3.6), or

$$t - t_0 \sim -\frac{\nu}{\gamma} \frac{\log(S_{kk}/S_0)}{S_{kk}}. \tag{3.9}$$

For arbitrary initial conditions within region (1), therefore, the solutions describe a system of transients, whereby the (nearly) inviscid form (3.9) is established.

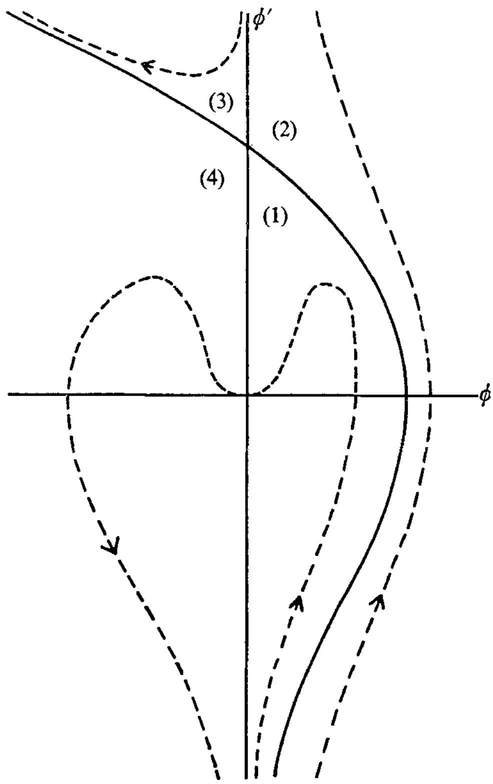

FIGURE 1

The asymptotic relevance of the inviscid approximation, for all initial conditions, follows from the fact that the Reynolds number $R = -S_{kk}^2/\nu\dot{S}_{kk}$ ultimately increases without limit, though logarithmically slowly.

When $\beta \neq 2$ solutions outside region (1) may again be rejected if S_{kk} is to decrease monotonically to zero as $t \to \infty$. When $\beta > 2$, the solutions are substantially the same as those sketched in figure 1, except that the nodal behaviour at the origin becomes modified to

$$\phi' \sim (\beta - 2)\phi, \tag{3.10}$$

which corresponds to the asymptotic behaviour

$$t - t_0 \sim C S_{kk}^{1-\beta} \tag{3.11}$$

as $t \to \infty$.

When $\beta < 2$, the situation is qualitatively different. The singularity at the origin separates into two: a saddle remaining at the origin; and a node moving to

$\phi = \frac{1}{6}(2-\beta)$, $\phi' = 0$, within region (1). The local solutions near this node are such that two special solutions have the form

$$\phi' = a\left(\phi - \frac{2-\beta}{6}\right), \quad \phi' = b\left(\phi - \frac{2-\beta}{6}\right), \quad 0 < a < b, \tag{3.12}$$

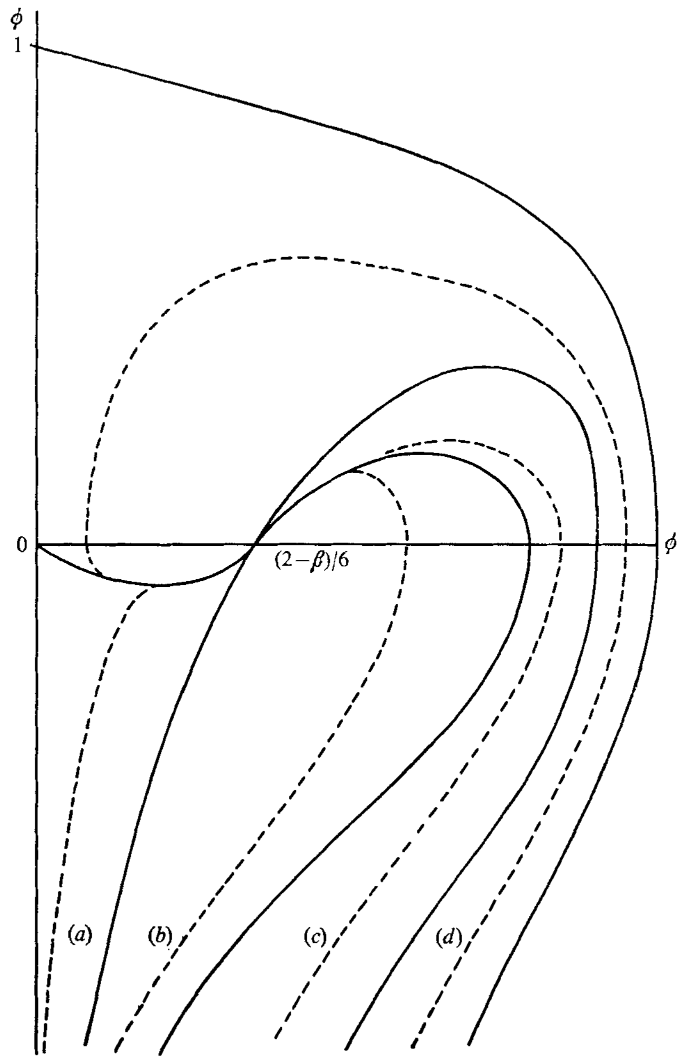

FIGURE 2

the first of which also passes through the singularity at the origin. All other solutions near the node have a high order of contact with this first solution. Thus, the former region (1) is split into four subregions by the solutions (3.12), in the manner sketched in figure 2.

All solutions in region (1) terminate at the node, corresponding to the asymptotic decay law

$$S_{kk} \sim \frac{6\nu}{(2-\beta)(t-t_0)}, \tag{3.13}$$

in which the decay Reynolds number R takes the universal value $6/(2-\beta)$. However, when the initial Reynolds number is large, this asymptotic state is not established directly through a system of rapid transients. The central role of the transients is to change R, without appreciable change in S_{kk}, from its 'forced' initial value to its 'natural' value at a point on the singular solution joining the origin to the node. If, at the end of this process, R is still large (subregions (*a*) and (*d*) in figure 2), the transient solution merges with the singular solution at a

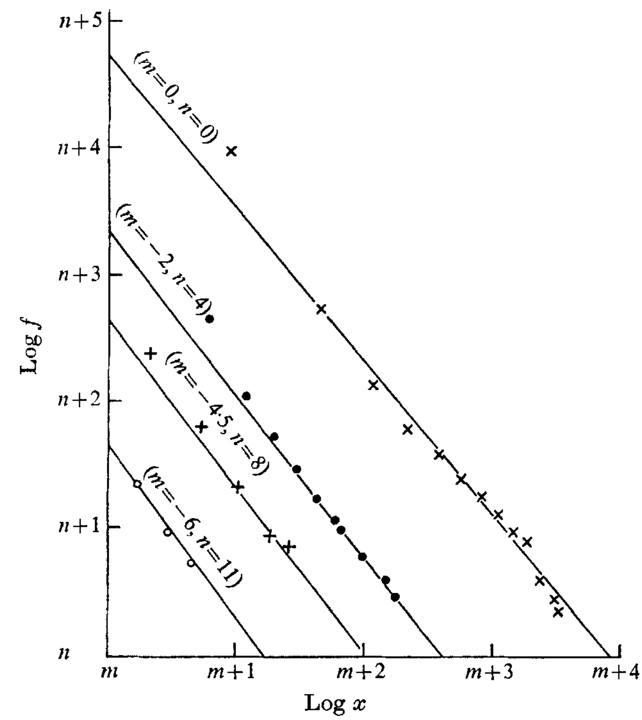

FIGURE 3. Comparison of the solutions of (3.3) with experiment, for $\beta = 1{\cdot}75$, $\gamma = 0{\cdot}5$: $\times$, $S_0 = 79{\cdot}5\ \mathrm{cm^2\,sec^{-2}}$; $\bullet$, $3{\cdot}10$; $+$, $0{\cdot}114$; $\bigcirc$, $0{\cdot}0120$. The scale S_0 is defined by the asymptotic behaviour $x \sim 32 f^{-\frac{3}{4}}$ as $x \to 0$.

point close to the origin, and the inviscid decay law (3.11) is established, in which state the system loses most of its energy. Unlike other ranges of β, however, when $\beta < 2$, the inviscid decay law (3.11) is not self-preserving; it contains its own mechanism for destruction, which is such that R slowly decreases to the value $6/(2-\beta)$.

The general nature of these results for $\beta < 2$ corresponds fairly closely with measurements of approximately isotropic turbulence in a wind-tunnel (see Batchelor & Townsend 1948). Evaluation of β and γ from such a comparison with experiment, however, is not at all sensitive. Clearly β cannot be substantially different from 2, in view of Batchelor & Townsend's approximate result

$$S_{kk} \propto (t-t_0)^{-1}, \tag{3.14}$$

but values as low as 1.75 seem to be acceptable. The theoretical results (ignoring the transients) for this value of β, and $\gamma = 0{\cdot}50$, are compared in figure 3 with the measurements of Batchelor & Townsend. The success of the comparison is partially dependent upon the fact that, for $\beta = 1{\cdot}75$, the asymptotic Reynolds number $6/(2-\beta)$ is 24, whereas the only experiments for which there are measurements over a reasonably extended period of decay are at Reynolds numbers in the range 50–150. By the standards of the theory, therefore, these Reynolds numbers are not large, and the average value of the exponent n in decay laws of the form

$$S_{kk} \propto (t-t_0)^{-n}$$

is significantly closer to 1 than its asymptotic value $\frac{4}{3}$ at infinite Reynolds number.

A more critical test of which values of β and γ, if any, can approximately represent the decay of isotropic turbulence would appear to await measurements over a more prolonged period of decay at significantly higher Reynolds number.

Appendix

1. *Form the constitutive relation for irrotational equilibrium*

For a steady, homogeneous, and irrotational, rate-of-strain field, the constitutive relation (2.13) becomes:

$$\begin{aligned}
p_1 e_{ij} e_{kl} S_{kl} S_{mm} &+ p_2 e_{ij} e_{kl} S_{km} S_{lm} \\
&+ p_3 e_{ik} e_{jk} S^2_{mm} + p_4 e_{ik} e_{jk} S_{lm} S_{lm} \\
&+ p_5 e_{il} e_{jk} S_{kl} S_{mm} + p_6 e_{il} e_{jk} S_{km} S_{lm} \\
&+ p_7 (e_{ik} S_{jk} + e_{jk} S_{ik}) e_{lm} S_{lm} + p_8 (e_{ik} S_{jl} + e_{jk} S_{il}) e_{kl} S_{mm} \\
&+ p_9 (e_{ik} S_{jl} + e_{jk} S_{il}) e_{km} S_{lm} + p_{10} (e_{ik} S_{jl} + e_{jk} S_{il}) e_{lm} S_{km} \\
&+ p_{11} S_{ik} S_{jk} e_{lm} e_{lm} + p_{12} S_{ik} S_{jl} e_{km} e_{lm} \\
&+ p_{13} S_{ij} S_{kk} e_{lm} e_{lm} + p_{14} S_{ij} S_{kl} e_{km} e_{lm} \\
&+ p_{15} e_{kl} e_{kl} S_{mp} S_{mp} \delta_{ij} + p_{16} e_{kl} e_{kl} S^2_{mm} \delta_{ij} \\
&+ p_{17} e_{kl} e_{km} S_{lm} S_{pp} \delta_{ij} + p_{18} e_{kl} e_{km} S_{lp} S_{mp} \delta_{ij} \\
&+ p_{19} e_{kl} e_{mp} S_{kl} S_{mp} \delta_{ij} + p_{20} e_{kl} e_{mp} S_{km} S_{lp} \delta_{ij} = 0,
\end{aligned}$$

where the fluid constants p_i are scalar invariants of p^3_8.

2. *The compatibility conditions for equilibrium*

The functions Q_i in (2.23) are:

$$\begin{aligned}
Q_1 &= q_1 g^2 + q_2 g + q_3 f^2 + q_4, \\
Q_2 &= q_5 fg + q_6 f, \\
Q_3 &= (-6q_1 + q_5 - q_3) g^2, \\
Q_4 &= (-\tfrac{1}{2}q_5 + q_3 + 2q_7) fg + (-3q_2 + \tfrac{1}{2}q_6 + q_8) f, \\
Q_5 &= (-3q_5 + 6q_3 + 6q_7) f^2 + q_7 g^2 + q_8 g, \\
Q_6 &= (3q_5 - 6q_3 + 6q_9) f^2 + q_9 g^2 + (-3q_2 + \tfrac{1}{2}q_6) g + q_{10}, \\
Q_7 &= (3q_5 - 6q_3 + 12q_9) fg,
\end{aligned}$$

where the fluid constants q_i are linear functions of the p_i.

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